Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 2, pp. 199–206, 2009

STRONG DISCONTINUITIES IN SPATIAL STATIONARY LONG-WAVE FLOWS OF AN IDEAL INCOMPRESSIBLE FLUID

A. K. Khe

UDC 533.6.011 + 517.948.34

Spatial stationary flows over an even bottom of a heavy ideal fluid with a free surface are considered. Jump relations for flows with a strong discontinuity are studied. It is shown that the flow parameters behind the jump are defined by a certain curve which is an analog of the (θ, p) diagram in gas dynamics. A shock polar and examples of flows with a hydraulic jump are constructed for a particular class of solutions.

Key words: *vortex shallow water, hydraulic jump, integrodifferential equations.*

Introduction. The long-wave approximation is widely used in the theoretical analysis of wave processes because, for real fluids, the asymptotics of solutions at large times is determined by long waves, due to the presence of viscosity. In addition, this approximation provides new results in analytical studies of nonlinear wave processes. The integrodifferential model of stationary ideal fluid flows studied in the present paper considers velocity shear along the vertical and is more complex than the classical shallow-water model.

Shear plane–parallel flows with discontinuities were studied in [1], and barotropic fluid flows in [2]. Generalized characteristics and hyperbolicity conditions for systems of integrodifferential shallow-water equations describing spatial stationary shear flows of a free-boundary ideal fluid layer were obtained in [3], and theory of spatial simple waves was constructed in [4].

The system of equations describing stationary long waves in spatial ideal fluid flows are not reduced to divergent form. It has been noted [5] that the flow equations can be rearranged so that the nondivergent terms are regular discontinuous functions. This allows one to consider solutions in the class of functions with a strong discontinuity and to derive jump relations from these equations. In the present paper, the indicated approach is used to study stationary hydraulic jumps. Relations linking the flow parameters behind the jump are found, and analogs of (θ, p) diagrams of two-dimensional gas dynamics are proposed. The problem of interaction of spatial shear flows is solved for a definite class.

1. Formulation of the Problem. We consider the long-wave equations describing spatial stationary flows of an ideal incompressible fluid in a gravity field:

$$uu_x + vu_y + wu_z + p_x/\rho = 0, \qquad uv_x + vv_y + wv_z + p_y/\rho = 0,$$
$$p_z = -\rho g, \qquad u_x + v_y + w_z = 0. \tag{1.1}$$

The equations are obtained from the exact Euler equations by asymptotic expansion in the small parameter $\varepsilon = H_0/L_0$ (H_0 and L_0 are the characteristic vertical and horizontal scales; it is assumed that $H_0 \ll L_0$). In system (1.1), u, v, and w are the fluid velocity components, p is the pressure, $\rho = \text{const}$ is the density, and x, y, and z are Cartesian coordinates in space.

Fluid flow is considered in a free-boundary layer $0 \leqslant z \leqslant h(x, y)$. On the free surface $z = h(x, y)$, the pressure is set constant: $p = p_0$. On the boundaries of the fluid layer, the kinematic conditions $w = 0$ at $z = 0$ and $w = uh_x + vh_y$ at $z = h$ should be satisfied.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; alekhe@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 2, pp. 37–45, March–April, 2009. Original article submitted October 9, 2007; revision submitted December 29, 2007.

To analyze the hyperbolic properties of the system, it is convenient to transform to mixed Eulerian-Lagrangian coordinates x', y', and λ by using the formulas [3]

$$x' = x, \qquad y' = y, \qquad \Phi(x', y', \lambda) = z.$$

Here the function $\Phi(x', y', \lambda)$ is a solution of the Cauchy problem

$$u(x, y, \Phi)\Phi_x + v(x, y, \Phi)\Phi_y = w(x, y, \Phi), \qquad \Phi\Big|_{x=0} = \Phi_0(y, \lambda).$$

It is assumed that $u(0, y, \Phi_0(y, \lambda)) \neq 0$, $\lambda \in [0, 1]$. The function $\Phi_0(y, \lambda)$ is chosen so that the value $\lambda = 0$ corresponds to the even bottom and $\lambda = 1$ to the free surface [$\Phi_0(y, 0) = 0$ and $\Phi_0(y, 1) = h(0, y)$]. In the new variables, the region occupied by the fluid corresponds to a fixed layer $\lambda \in [0, 1]$ and the equations for spatial stationary flows become

$$uu_x + vu_y + gh_x = 0, \qquad uv_x + vv_y + gh_y = 0, \qquad (uH)_x + (vH)_y = 0. \tag{1.2}$$

Here the new required function $H(x, y, \lambda) = \Phi_\lambda(x, y, \lambda)$ is introduced, which is related to the fluid layer thickness h by the formula $h = \int\limits_0^1 H\, d\lambda$.

System (1.2) is integrodifferential, and the unknowns are the functions u, v, and H, which depend on the variables x, y, λ. In [6] (see also [7]), hyperbolicity theory for the differential equations is extended to the case of differential equations with operator coefficients. The generalized characteristics are some curves in the plane of variables (x, y), and the variable λ is treated as a parameter.

It has been shown [3] that system (1.2) has two families of characteristics. We use $\boldsymbol{u}$ to denote the projection of the velocity onto the horizontal plane: $\boldsymbol{u} = (u, v)$. Let $q(x, y, \lambda)$ and $\theta(x, y, \lambda)$ be the modulus and slope of the vector $\boldsymbol{u}$: $u = q\cos\theta$ and $v = q\sin\theta$. Then, the generalized characteristics of the continuous spectrum are the particle trajectories [3]: $dy/dx = \tan\theta\Big|_{\lambda=\nu}$, $\nu \in (0, 1)$, and the characteristics of the discrete spectrum $dy/dx = \tan\gamma^i$ ($i = 1, 2$) are defined by the characteristic equation

$$\chi(\gamma) \equiv 1 - g\int\limits_0^1 \frac{H\, d\lambda}{q^2 \sin^2(\theta - \gamma)} = 0. \tag{1.3}$$

The function $\chi(\gamma)$ is π-periodic, convex upward, and defined on the set of intervals $[\max_\lambda \theta + \pi k, \min_\lambda \theta + \pi(k+1)]$; k is an integer.

2. Flows with Discontinuities. In the present section, use is made of the discontinuity relations obtained in [5] for the equations of stationary flows (1.2). It is shown that the set of states behind the jump is determined by some curve similar to the (θ, p)-polar curve in gas dynamics.

Let us consider flows with strong discontinuities concentrated on a cylindrical surface Γ given by the equation $S(x, y) = 0$ in the space of variables (x, y, λ). We next speak of the a discontinuity line in the plane (x, y) [the orthogonal projection of Γ onto the plane (x, y)] and assume that λ is a parameter. System (1.2) cannot be reduced to divergent form. In [5], it is proposed to use the following form of system (1.2), which is equivalent to the initial one for smooth solutions:

$$\Big(\frac{u^2 + v^2}{2}\Big)_{\lambda x} - (v(v_x - u_y))_\lambda = 0, \qquad \Big(\frac{u^2 + v^2}{2}\Big)_{\lambda y} + (u(v_x - u_y))_\lambda = 0,$$

$$\Big(\int\limits_0^1 Hu^2\, d\lambda + \frac{gh^2}{2}\Big)_x + \Big(\int\limits_0^1 Huv\, d\lambda\Big)_y = 0, \qquad \Big(\int\limits_0^1 Huv\, d\lambda\Big)_x + \Big(\int\limits_0^1 Hv^2\, d\lambda + \frac{gh^2}{2}\Big)_y = 0. \tag{2.1}$$

We consider the class of solutions of system (2.1) in which the functions u, v, and H and the vertical component of the vortex vector $v_x - u_y$ and their derivatives with respect to the variable λ are piecewise smooth functions which have a discontinuity of the first order on the surface Γ. It has been shown [5] that in this class of functions, the notion of a solution with a strong discontinuity is defined correctly [i.e., Eqs. (2.1) are meaningful in the class of generalized functions] and system (2.1) defines discontinuity relations.

Let us introduce the tangent $\boldsymbol{\tau}$ and normal $\boldsymbol{n}$ vectors to the discontinuity line Γ. The tangent and normal components of the vector $\boldsymbol{u}$ will be denoted by $u_\tau = \boldsymbol{u}\cdot\boldsymbol{\tau}$ and $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$, respectively. Square brackets denote the jump of the function across the discontinuity: $[\varphi] = \varphi_2 - \varphi_1$ (subscript 1 corresponds to the state ahead of the discontinuity front; subscript 2 to the state behind the front). Strong-discontinuity relations for the model of spatial shear flows were derived in [5]. In the case of stationary flows, these relations become

$$[u_\tau] = 0, \qquad [Hu_n] = 0, \qquad [(u_n^2)_\lambda] = 0; \tag{2.2}$$

$$\Big[\int_0^1 Hu_n^2\,d\lambda + \frac{gh^2}{2}\Big] = 0; \tag{2.3}$$

$$\Big[\frac{u_n^2}{2} + gh\Big] \leqslant 0. \tag{2.4}$$

System (2.2)–(2.4) links the flow parameters on both sides of the discontinuity front and the normal components to the front. It has also been proved [5] that the flow parameters behind the front of the jump are determined uniquely, given the flow parameters ahead of the jump, the velocity of the front, and the normal vector to the jump front.

We note that, in the stationary case, the general stability conditions for the discontinuity lead to the flow supercriticality condition ahead of the jump, namely: for the angle α of inclination of the jump to the x axis, the following condition should be satisfied [5]:

$$\chi_1(\alpha) > 0,$$

where $\chi_1(\alpha)$ is the value of the characteristic function (1.3) in the incoming flow:

$$\chi_1(\alpha) = 1 - g\int_0^1 \frac{H_1\,d\lambda}{q_1^2\sin^2(\theta_1-\alpha)}.$$

If γ^1 and γ^2 are the angles of inclination of the characteristics to the x axis [roots of the function $\chi_1(\gamma)$], then, due to the convexity of the function $\chi_1(\gamma)$, the angle α should be in the interval $\gamma^1 < \alpha < \gamma^2$. This condition is similar to the following condition: in the case of a two-dimensional stationary gas flow, the angle of inclination of an oblique shock to the incoming-flow velocity should be larger than the Mach angle.

In studies of discontinuity relations for the stationary case, it is convenient to use polar coordinates in the hodograph plane. From relations (2.2) and (2.3), it follows that the flow parameters behind the jump front are given by the formulas

$$q_2 = \sqrt{q_1^2 - K}, \qquad \cos(\theta_2-\alpha) = \frac{q_1\cos(\theta_1-\alpha)}{\sqrt{q_1^2-K}}; \tag{2.5}$$

$$H_2 = \frac{H_1 q_1 \sin(\theta_1-\alpha)}{\sqrt{q_1^2\sin^2(\theta_1-\alpha) - K}}, \tag{2.6}$$

where K is an unknown quantity independent of λ. The dependence of K on the angle α is determined from the discontinuity relation

$$\int_0^1 H_1q_1\sin(\theta_1-\alpha)\sqrt{q_1^2\sin^2(\theta_1-\alpha)-K}\,d\lambda + \frac{g}{2}\Big(\int_0^1 \frac{H_1q_1\sin(\theta_1-\alpha)}{\sqrt{q_1^2\sin^2(\theta_1-\alpha)-K}}\,d\lambda\Big)^2$$

$$= \int_0^1 H_1\,q_1^2\sin^2(\theta_1-\alpha)\,d\lambda + \frac{g}{2}\Big(\int_0^1 H_1\,d\lambda\Big)^2.$$

This equation is investigated in detail in [5].

The flow parameters behind the jump front are found from Eqs. (2.5) and (2.6) with the equation for K taken into account. The dependence $K(\alpha)$ is determined parametrically and the angle α defining the angle of inclination of the jump is chosen as a parameter.

Let θ_2^0 be the angle of inclination of the velocity to the x axis for $\lambda = 0$ that satisfies the relation

$$\cos(\theta_2^0 - \alpha) = \frac{q_1^0 \cos(\theta_1^0 - \alpha)}{\sqrt{(q_1^0)^2 - K(\alpha)}}. \tag{2.7}$$

Integration of Eq. (2.6) yields the following expression for the fluid layer depth behind the jump

$$h_2 = \int_0^1 \frac{H_1 q_1 \sin(\theta_1 - \alpha)}{\sqrt{q_1^2 \sin^2(\theta_1 - \alpha) - K(\alpha)}}\, d\lambda. \tag{2.8}$$

The curve in the plane (θ_2^0, h_2) given parametrically by Eqs. (2.7) and (2.8), defines the possible states behind the hydraulic jump and is an analog of (θ, p) diagrams in stationary two-dimensional gas dynamics. We note that if one of the parameters (θ_2^0 or h_2) is known, then, by inverting dependence (2.7) or (2.8), it is possible to determine the angle α and then, using formulas (2.5) and (2.6), the functions q_2, θ_2, and H_2.

Thus, the set of states behind the different jumps resulting from some fixed flow, forms a one-parameter family. As the parameter, one can use the angle α of inclination of the jump to the x axis, the flow depth behind the jump h_2 or the angle θ_2^0 between the velocity on the bottom and the x axis.

3. Flows with Velocity Modulus Constant Along the Depth. Below, we study the class of flows with velocity modulus constant along the depth. Equations are derived that allow one to determine the angle α of inclination of the jump for a known direction of the velocity behind the jump on the bottom, and (θ^0, h) diagrams are constructed.

Previously, it has been shown [3] that system (1.2) has particular solutions satisfying the conditions

$$q_\lambda = 0, \qquad \theta_\lambda / H = A \qquad (A = \text{const}). \tag{3.1}$$

The equality $\theta_\lambda/H = A$ implies that the angle θ depends linearly on the Eulerian variable z:

$$\theta(z) = \theta^0 + A \int_0^\lambda \Phi_{\lambda_1}(x, y, \lambda_1)\, d\lambda_1 = \theta^0 + Az$$

(θ^0 is the value of θ on the bottom for $\lambda = 0$).

We note that conditions (3.1) are conserved across the jump. Indeed, the jump relations (2.2) are written in the variables q and θ as follows:

$$q_1 \cos(\alpha - \theta_1) = q_2 \cos(\alpha - \theta_2),$$

$$H_1 q_1 \sin(\alpha - \theta_1) = H_2 q_2 \sin(\alpha - \theta_2), \qquad (q_1^2)_\lambda = (q_2^2)_\lambda.$$

From the last equation, we obtain $q_{2\lambda} = 0$, and from the first two equations, we have $\theta_{2\lambda}/H_2 = \theta_{1\lambda}/H_1 = A$.

Thus, if the flow ahead of the front satisfies conditions (3.1), the flow behind the strong-discontinuity front belongs to the class of solutions that also satisfy conditions (3.1). At the discontinuity, the following three relations [following from (2.2) and (2.3)] linking the four parameters q_2, θ_2^0, h_2, and α should be satisfied:

$$q_1 \cos(\alpha - \theta_1^0) = q_2 \cos(\alpha - \theta_2^0), \qquad q_1 \cos(\alpha - \theta_1^1) = q_2 \cos(\alpha - \theta_2^1); \tag{3.2}$$

$$q_1^2 h_1 + \frac{q_1^2}{2A}\Big(\sin 2(\alpha - \theta_1^1) - \sin 2(\alpha - \theta_1^0)\Big) + g h_1^2$$

$$= q_2^2 h_2 + \frac{q_2^2}{2A}\Big(\sin 2(\alpha - \theta_2^1) - \sin 2(\alpha - \theta_2^0)\Big) + g h_2^2 \tag{3.3}$$

$$\Big(A h_1 = \theta_1^1 - \theta_1^0, \qquad A h_2 = \theta_2^1 - \theta_2^0, \qquad \theta_i^0 = \theta_i\Big|_{\lambda=0}, \qquad \theta_i^1 = \theta_i\Big|_{\lambda=1}\Big).$$

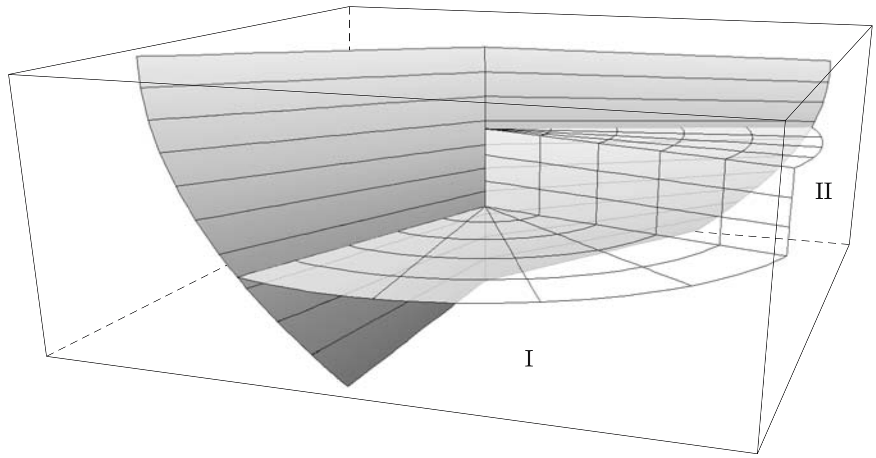

Fig. 1. Hydraulic jump: region ahead of the jump (I) and region behind the jump (II).

Figure 1 gives an example of flow with a hydraulic jump along inclined walls which have the shape of enveloping surfaces. Particles move from region I to region II, in each of which their motion is horizontal and parallel to the generator of the wall at the same height.

Let the parameters of the incoming flow satisfy (3.1). We derive equations to determine the angle of inclination of the jump and the flow parameters behind the jump from the known angle θ_2^0. The jump relations (3.2) lead to

$$q_2 = q_1 \frac{\cos(\alpha - \theta_1^0)}{\cos(\alpha - \theta_2^0)}, \qquad h_2 = A^{-1}\Big(\alpha - \theta_2^0 - \arccos\frac{\cos(\alpha - \theta_1^1)\cos(\alpha - \theta_2^0)}{\cos(\alpha - \theta_1^0)}\Big).$$

Substitution of these expressions into relation (3.3) yields the following equation for the function $\alpha(\theta_2^0)$:

$$A^{-1}q_1^2\Big(Ah_1 - \sin 2(\alpha - \theta_1^0)\sin^2 Ah_1 - \frac{1}{2}\cos 2(\alpha - \theta_1^0)\sin 2Ah_1\Big) + gh_1^2$$

$$= A^{-1}q_2^2(\alpha, \theta_2^0)\Big(Ah_2 - \cos(\alpha - \theta_2^0)\sin(\alpha - \theta_2^0)$$

$$+ \sqrt{1 - \Big(\frac{\cos(\alpha - \theta_1^1)\cos(\alpha - \theta_2^0)}{\cos(\alpha - \theta_1^0)}\Big)^2}\,\frac{\cos(\alpha - \theta_1^1)\cos(\alpha - \theta_2^0)}{\cos(\alpha - \theta_1^0)}\Big) + gh_2^2(\alpha, \theta_2^0).$$

To determine the flow parameters behind the discontinuity, by analogy with the theory of oblique shocks in two-dimensional gas dynamics [8], it is possible to use the hodograph of the jumps. For the flows considered, an analog of the (θ, p) polars used in stationary two-dimensional gas dynamics [8] are diagrams constructed in the plane of variables (θ^0, h).

We choose the coordinate system so that the equality $\theta_1^0 = 0$ is satisfied. From the discontinuity relations (3.2), we have

$$q_1 \cos\alpha = q_2 \cos(\alpha - \theta_2^0), \qquad q_1 \cos(\alpha - Ah_1) = q_2 \cos(\alpha - \theta_2^0 - Ah_2).$$

From the first relation, q_2 is expressed as

$$q_2 = q_1/(\cos\theta_2^0 + \sin\theta_2^0 \tan\alpha).$$

The second relation divided by the first is transformed to a quadratic equation for $\tan\alpha$:

$$\sin Ah_1 \tan\theta_2^0 \tan^2\alpha + (\cos Ah_1 \tan\theta_2^0 + \sin Ah_1 - \cos Ah_2 \tan\theta_2^0 - \sin Ah_2)\tan\alpha$$

$$+ \cos Ah_1 - \cos Ah_2 + \sin Ah_2 \tan\theta_2^0 = 0. \tag{3.4}$$

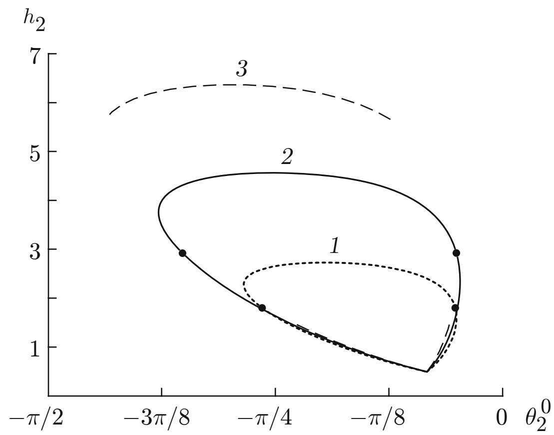

Fig. 2. (θ^0, h) Diagram for $q_1 = 2$ (1), 2.5 (2), and 2.9 (3).

Relation (3.3) can be rearranged to

$$(q_1^2 h_1 + g h_1^2)(1 + \tan^2 \alpha) + \frac{q_1^2}{2A}\Big(2\tan\alpha(\cos 2Ah_1 - 1) + (\tan^2\alpha - 1)\sin 2Ah_1\Big)$$

$$= (q_2^2 h_2 + g h_2^2)(1 + \tan^2 \alpha) + \frac{q_2^2}{2A}\Big(2\tan\alpha\Big(\cos 2(\theta_2^0 + Ah_2) - \cos 2\theta_2^0\Big)$$

$$+ (\tan^2\alpha - 1)\Big(\sin 2(\theta_2^0 + Ah_2) - \sin 2\theta_2^0\Big)\Big). \qquad (3.5)$$

Then, expressing $\tan\alpha$ from Eq. (3.4):

$$\tan\alpha = \frac{-\cos Ah_1 \tan\theta_2^0 - \sin Ah_1 + \cos Ah_2 \tan\theta_2^0 + \sin Ah_2}{2\sin Ah_1 \tan\theta_2^0}$$

$$\pm \Big((\cos Ah_1 \tan\theta_2^0 + \sin Ah_1 - \cos Ah_2 \tan\theta_2^0 - \sin Ah_2)^2$$

$$- 4\sin Ah_1 \tan\theta_2^0(\cos Ah_1 - \cos Ah_2 + \sin Ah_2 \tan\theta_2^0)\Big)^{1/2} \Big/ (2\sin Ah_1 \tan\theta_2^0) \qquad (3.6)$$

and substituting the obtained expression into (3.5), we obtain the required equation relating the quantities θ_2^0 and h_2:

$$P(\theta_2^0, h_2) = 0.$$

The plus and minus signs in the expression for $\tan\alpha$ in (3.6) correspond to the waves moving to the right or to the left.

Figure 2 shows three polars obtained for various values of q_1 in the plane of variables (θ_2^0, h_2). The points correspond to the states at which the flow behind the jump changes from the supercritical to a subcritical state. For $q_1 = 2.9$, the curve is discontinuous (two-connected), which is due to the absence of a solution in the examined class of flows (the discontinuity relations and their resolvability are investigated in [5]).

4. Shear-Flow Interaction. In the class of flows defined by conditions (3.1), we construct the solution of the problem of shear-flow interaction. We assume that two flows with parameters q_1, $\theta_1(\lambda)$ and q_2, $\theta_2(\lambda)$ satisfying relations (3.1) with the same constant A occupy regions I and II (Fig. 3). Let the velocity directions satisfy the condition $\theta_2(0) > \theta_1(0)$, which implies that the flows impinge against each other. It is required to construct the flow in the flow-interaction region.

Because the value of the constant A does not change across the jump, the solution of the flow interaction problem can be sought in the class of discontinuous flows (3.1). For this, we construct (θ^0, h) diagrams (Fig. 4) from the points B and C corresponding to the flow regions I and II. These curves intersect each other, generally speaking,

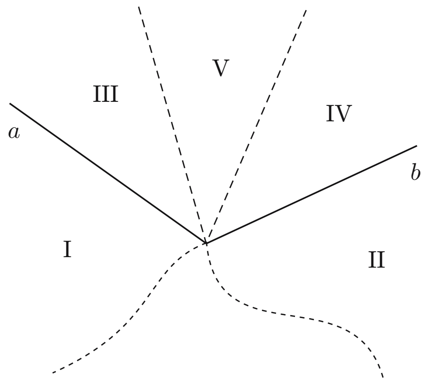

Fig. 3

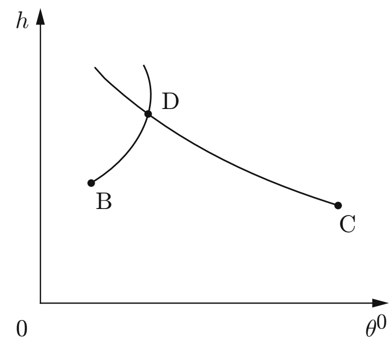

Fig. 4

Fig. 3. Interaction of shear flows: I and II are the incoming-flow regions; III and IV are the flow regions behind the jumps; V is the region of conjugation between the flows in regions III and IV; a and b are hydraulic jumps.

Fig. 4. Solution of the problem of flow interaction: point B refers to the flow region I, point C to the flow region II, and point D to the flow regions III–V.

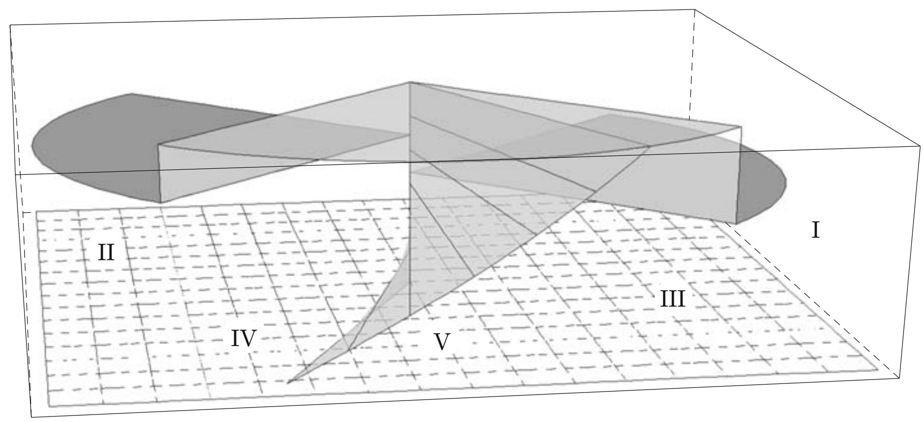

Fig. 5. Interaction of shear flows: I and II are the flow regions before interaction; III and IV are the flow regions behind the jump; V is the flow conjugation region.

at two points. In the gas-dynamic theory of oblique shocks, it is common to choose a state that corresponds to the lower point of intersection of the diagrams [9]; therefore, in this case, we also choose the point D that corresponds to the smaller value of h.

The point D of intersection of the diagrams determines the angle of inclination of the velocity on the bottom θ^0 and the fluid-layer depth after the passage of the flows through the jumps (the flows in the regions III and IV in Fig. 3). Because the velocity direction on the bottom and the constant A in regions III and IV are identical, the flow velocity directions behind the jump also are identical:

$$\theta = \theta_3 = \theta_4 = \theta_3^0 + Az.$$

However, the quantities q_3 and q_4, generally speaking, are different; therefore, the flows in regions III and IV are conjugate in zone V along the contact discontinuity having the shape of the enveloping surface $\theta = \theta_3^0 + Az$. Figure 5 shows an example of the flow configuration resulting from the interaction of the shear flows.

I am grateful to V. M. Teshukov for attention to this work.

This work was supported by the Russian Foundation for Basic Research (Grant No. 07-01-00609), the International INTAS foundation (Grant No. 05-109-4107), and the Program for Support of Leading Scientific Schools of the Russian Federation (Grant No.NSh-2826.2008.1).

REFERENCES

1. V. M. Teshukov, "Hydraulic jump in the shear flow of an ideal incompressible fluid," *J. Appl. Mech. Tech. Phys.*, **36**, No. 1, 10–18 (1995).
2. V. M. Teshukov, "Hydraulic jump in shear flow of a barotropic fluid," *J. Appl. Mech. Tech. Phys.*, **37**, No. 5, 675–682 (1996).
3. V. M. Teshukov, "Spatial stationary long waves in shear flows," *J. Appl. Mech. Tech. Phys.*, **45**, No. 2, 172–180 (2004).
4. V. M. Teshukov, "Spatial simple waves on a shear flow," *J. Appl. Mech. Tech. Phys.*, **43**, No. 5, 661–670 (2002).
5. V. M. Teshukov and A. K. Khe, "Model of a strong discontinuity for the equations of spatial long waves propagating in a free-boundary shear flow," *J. Appl. Mech. Tech. Phys.*, **49**, No. 4, 693–698 (2008).
6. V. M. Teshukov, "Hyperbolicity of the long-wave equations," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–559 (1985).
7. V. Yu. Liapidevskii and V. M. Teshukov, *Mathematical Models of Long-Wave Propagation in an Inhomogeneous Fluid* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (2000).
8. L. V. Ovsyannikov, *Lectures in the Foundations of Gas Dynamics* [in Russian], Nauka, Moscow (1981).
9. V. M. Men'shchikov and V. M. Teshukov, *Gas Dynamics. Problems and Exercises: Manual* [in Russian], Novosib. Gos. Univ., Novosibirsk (1990).